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A generalization of Barbashin–Krasovski theorem

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Abstract

The classical criterion of asymptotic stability of the zero solution of equations $x' = f(t,x)$ is that there exists a function $V(t,x)$, $a(\|x\|) \leqslant V(t,x) \leqslant b(\|x\|)$ for some $a, b \in K$, such that $\dot V(t,x) \leqslant -c(\|x\|)$ for some $c \in K$. In this paper we prove that if $f(t,x)$ is bounded, $\dot V(t,x)$ is uniformly continuous and bounded, then the condition that $\dot V(t,x) \leqslant -c(\|x\|)$ can be weakened and replaced by $\dot V(t,x) \leqslant 0$ and $\{(t,x)\colon\ x \neq 0,\ \dot{\bar V}(t,x) = 0\}$ contains no complete trajectory of $x' = \bar f(t,x)$, $t \in [-T,T]$, where $\bar V(t,x) = \lim_{k\to\infty} V(t+t_k,x)$, $\bar f(t,x) = \lim_{k\to\infty} f(t+t_k,x)$ uniformly for $(t,x) \in [-T,T] \times B_H$.

Keywords: Asymptotic stability; Lyapunov's direct method; Barbashin–Krasovski theorem

Consider a system of ordinary differential equations

$$\frac{dx}{dt} = f(t,x), \tag{1}$$

where $x \in B_H$, $B_H = \{x \in R^n\colon\ \|x\| < H\}$; the function $f : [0,+\infty) \times B_H \to R$ is smooth enough to ensure existence and uniqueness of the solution of the initial value problem associated with (1); $f(t,0) \equiv 0$, $t \geqslant 0$.

The classical criterion of asymptotic stability of the zero solution of equations $x' = f(t,x)$, which was obtained by Lyapunov [1], is that there exists a positive definite function V which has infinitesimal upper bound such that $\dot V(t,x)$ is negative definite (i.e., $\dot V(t,x) \leqslant -c(\|x\|)$

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for some $c \in K$). In applications one often constructs a positive definite V which derivative is not negative definite, but is less than or equal to zero. Exactly for such cases Barbashin and Krasovski created an asymptotic stability criterion and proved that if f and V are periodic, then the condition that $\dot{V}(t,x) \leqslant -c(\|x\|)$ can be weakened and replaced by $\dot{V}(t,x) \leqslant 0$ and $\{(t,x)\colon\ x \neq 0,\ \dot{V}(t,x) = 0\}$ contains no complete trajectory of (1) [1–3]. Later, Ignatyev extended this criterion for the case where f is almost periodic [4]. Recently, much progress has been made concerning the criterion of asymptotic stability, see [5–9]. In [5,6], criteria are obtained under the assumption that V is a positive definite function and $\dot{V}(t,x)$ is a nonpositive function, in [7] the conditions imposed on the function V are also weakened under certain circumstances. In this paper, we prove that for nonautonomous system (1), if $f(t,x)$ is bounded and there exist a function $V(t,x)$, a sequence $\{t_k\}$ with $t_k \to +\infty$ $(k \to \infty)$, a $T > 0$, such that $a(\|x\|) \leqslant V(t,x) \leqslant b(\|x\|)$ for some $a, b \in K$, $\dot{V}(t,x) \leqslant 0$, $\dot{V}(t,x)$ is uniformly continuous and bounded, $\lim_{k\to\infty} V(t+t_k,x) = \bar{V}(t,x)$, $\lim_{k\to\infty} f(t+t_k,x) = \bar{f}(t,x)$ uniformly for $(t,x) \in [-T,T] \times B_H$ and $\{(t,x)\colon\ x \neq 0,\ \dot{\bar{V}}(t,x) = 0\}$ contains no complete trajectory of $x' = \bar{f}(t,x)$, $t \in [-T,T]$, then the zero solution is asymptotically stable.

First, we introduce a simple lemma, the proof of which can be found in [5, Lemma 2].

Lemma. *Consider a C^1 function $f\colon [0,+\infty) \to R$. If $f(x)$ satisfies*:

(i) *the limit* $\lim_{x\to+\infty} f(x)$ *exists,*
(ii) *$f'(x)$ is uniformly continuous on* $[0,+\infty)$,

then $\lim_{t\to+\infty} f'(x) = 0$.

Now we give the main result of this paper.

Theorem. *Consider differential equations* (1) *with the corresponding hypothesis. Besides we assume that $f(t,x)$ is uniformly continuous and bounded on* $[0,+\infty) \times B_H$. *Suppose that there exists a uniformly continuous and bounded C^1 function $V(t,x)\colon [0,+\infty) \times B_H \to R$ such that for* $(t,x) \in [0,+\infty) \times B_H$:

(i) *$a(\|x\|) \leqslant V(t,x) \leqslant b(\|x\|)$, where $a, b \in K$; K is the class of Hahn's functions* [1];
(ii) *$\dot{V}(t,x) \leqslant 0$, where $\dot{V}(t,x) = \frac{\partial V(t,x)}{\partial t} + (\frac{\partial V(t,x)}{\partial x}, f(t,x))$;*
(iii) *$\dot{V}(t,x)$ is uniformly continuous and bounded.*

Ascoli–Arzelà theorem assures that there exist a sequence $\{t_k\}$ with $t_k \to +\infty$ $(k \to \infty)$, a positive number T, a continuous function $\bar{f}(t,x)\colon [-T,T] \times B_H \to R$ and a C^1 function $\bar{V}(t,x)\colon [-T,T] \times B_H \to R$ such that $\lim_{k\to\infty} V(t+t_k,x) = \bar{V}(t,x)$, $\lim_{k\to\infty} f(t+t_k,x) = \bar{f}(t,x)$ *uniformly with respect to* $(t,x) \in [-T,T] \times B_H$.

Moreover, if

(iv) *except for $x = 0$, $M = \{(t,x) \in [-T,T] \times B_H\colon\ \dot{\bar{V}}(t,x) = 0\}$ contains no complete trajectory of*

$$\frac{dx}{dt} = \bar{f}(t,x), \qquad t \in [-T,T], \tag{2}$$

where $\dot{\bar{V}}(t,x) = \frac{\partial \bar{V}(t,x)}{\partial t} + (\frac{\partial \bar{V}(t,x)}{\partial x}, f(t,x))$, *then the solution*

$$x = 0$$

of differential equations (1) *is asymptotically stable.*

If (i) *is replaced by*

(i)′ *for some* $t_0 > 0$ *and each* $\delta > 0$, *there exists* $x_0 \in B_\delta$, *such that* $V(t_0, x_0) < 0$; $V(t, 0) = 0$,

then the solution $x = 0$ *is unstable.*

Proof. To prove the first part of this statement, obverse first that the zero solution $x = 0$ is uniformly stable by conditions (i) and (ii). Therefore for any $t_0 > 0$ and $h \in (0, H)$ there exists $\delta > 0$ such that any solution $x(t)$ of Eq. (1) satisfies $\|x(t)\| < h$ for every $t > t_0$, if $\|x(t_0)\| < \delta$. Choosing such $\delta > 0$, we will show that any solution $x(t)$ with $\|x(t_0)\| < \delta$ satisfies

$$\lim_{t\to+\infty} x(t) = 0.$$

If this were not true, due to uniform stability, there would be a solution $X(t)$ and $\eta > 0$ such that

$$\eta \leqslant \|X(t)\| \leqslant h, \quad \text{for } t \geqslant 0.$$

Due to the boundedness of $f(t,x)$, $X(t)$ is uniformly continuous on $[0, +\infty)$. Thus, $\{x_k(t)\}$ is equicontinuous and uniformly bounded on $[-T, T]$, where $x_k(t) = X(t + t_k)$, $t \in [-T, T]$. It follows from Ascoli–Arzelà theorem that $\{x_k(t)\}$ contains a subsequence $\{x_{k_i}(t)\}$ and there exists a C^1 function $\bar{x}$ such that

$$\lim_{i\to\infty} x_{k_i}(t) = \bar{x}(t)$$

uniformly with respect to $t \in [-T, T]$. But $x_{k_i}(t)$ is a solution of $x' = f(t + t_{k_i}, x)$, $t \in [-T, T]$, and $\lim_{i\to\infty} f(t + t_{k_i}, x) = \bar{f}(t, x)$ uniformly for $(t, x) \in [-T, T] \times B_H$. Therefore $\bar{x}(t)$ is a solution of (2). Let $v(t) = V(t, x(t))$, $\bar{v}(t) = \bar{V}(t, \bar{x}(t))$, we have

$$\lim_{i\to\infty} v(t + t_{k_i}) = \bar{v}(t), \quad t \in [-T, T]. \tag{3}$$

By condition (iii), $\{v'(t + t_{k_i})\}_{i=1}^{\infty}$ is equicontinuous and uniformly bounded on $[-T, T]$, so $\{t_{k_i}\}$ contains a subsequence $\{\tau_j\}$ such that $\{v'(t + \tau_j)\}$ converges uniformly. From (3), we derive that

$$\lim_{j\to\infty} v'(t + \tau_j) = \bar{v}'(t), \quad t \in [-T, T].$$

By condition (iv), there exists $t_1 \in [-T, T]$, such that $\bar{v}'(t_1) \neq 0$, therefore

$$\lim_{j\to\infty} v'(t_1 + \tau_j) \neq 0. \tag{4}$$

But from condition (i) we have $v(t) \geqslant 0$, from condition (ii) we know that $v(t)$ is monotonically nonincreasing. Therefore the limit $\lim_{t\to+\infty} v(t)$ exists. From condition (iii) we know that $v'(t)$ is uniformly continuous on $[0, +\infty)$. It follows from Lemma that

$$\lim_{t\to+\infty} v'(t) = 0,$$

this equality contracts (4), so the solution $x = 0$ is asymptotically stable.

The proof of the thesis concerning instability can be proved in the same spirit, and we omit the details. This completes the proof of Theorem. □

The following example illustrates the usage of Theorem.

Example. Let $f(y)$ and $g(x)$ be continuous functions from $R \to R$ and $h(t,x,y)$ be a uniformly continuous function from $R^3 \to R$. Consider the equivalent form of the Lienard type equation

$$\dot{x} = y, \qquad \dot{y} = -h(t,x,y)y - f(y)g(x),$$

the origin is asymptotically stable if

(i) there exist a continuous function $k(x,y)$, a sequence $\{t_k\}$ with $t_k \to +\infty$ $(k \to \infty)$ and $c > 0$, such that $0 \leqslant h(t,x,y) \leqslant k(x,y)$ and $h(t_k,x,y) \geqslant c$;
(ii) $f(y) > 0$, for every y;
(iii) $g(x)x > 0$ for every $x \neq 0$ and $g(0) = 0$.

In fact, using $V(x,y) = \int_0^y \frac{u\,du}{f(u)} + \int_0^x g(u)\,du$, we have

$$\dot{V}(x,y) = -y^2 h(t,x,y)/f(y) \leqslant 0.$$

By uniform continuity and boundedness of $h(t,x,y)$ on $[0,+\infty) \times B_H$ and Ascoli–Arzelà theorem, there exist $T > 0$, $\{t_{k_i}\}$ (a subsequence of $\{t_k\}$) and a continuous function $\bar{h}(t,x,y)$ such that

$$\lim_{i\to\infty} h(t + t_{k_i}, x, y) = \bar{h}(t,x,y), \quad \text{uniformly for } (t,x,y) \in [-T,T] \times B_H.$$

Obviously, $M := \{(t,x,y) \in [-T,T] \times B_H\colon\ \dot{V}(t,x,y) = 0\} = \{(t,x,y) \in [-T,T] \times B_H\colon y = 0\}$ contains no complete trajectory of

$$\dot{x} = y, \qquad \dot{y} = -\bar{h}(t,x,y)y - f(y)g(x), \quad (t,x,y) \in [-T,T] \times B_H,$$

except for $x = y = 0$. Therefore by Theorem the zero solution is asymptotically stable.

References

[1] N. Rouche, P. Habets, M. Laloy, Stability Theory by Liapunov's Direct Method, Springer-Verlag, New York, 1977.
[2] E.A. Barbashin, N.N. Krasovski, On the global stability of motion, Soviet Math. Dokl. 86 (1952) 453–456.
[3] N.N. Krasovski, Stability of Motion, Stanford Univ. Press, Stanford, CA, 1963.
[4] A.O. Ignatyev, On the stability of equilibrium for almost periodic systems, Nonlinear Anal. 29 (8) (1997) 957–962.
[5] Liangping Jiang, On the asymptotic stability of equilibrium, J. Math. Anal. Appl. 301 (2005) 378–383.
[6] A.O. Ignatyev, On the asymptotic stability in functional differential equations, Proc. Amer. Math. Soc. 127 (1999) 1753–1760.
[7] A.O. Ignatyev, Investigation of stability by using Lyapunov functions of fixed sign, Ukrainian Math. Bull. 2 (2005) 77–86.
[8] A.O. Ignatyev, On the partial equiasymptotic stability in functional differential equations, J. Math. Anal. Appl. 268 (2) (2002) 615–628.
[9] Alexander O. Ignatyev, Oleksiy A. Ignatyev, On the stability in periodic and almost periodic difference systems, J. Math. Anal. Appl. 313 (2) (2006) 678–688.